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Inclusions and inhomogeneities in strain gradient elasticity with couple stresses and related problems

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Abstract

At small length scales and/or in presence of large field gradients, the implicit long wavelength assumption of classical elasticity breaks down. Postulating a form of second gradient elasticity with couple stresses as a suitable phenomenological model for small-scale elastic phenomena, we herein extend Eshelby's classical formulation for inclusions and inhomogeneities. While the modified size-dependent Eshelby's tensor and hence the complete elastic state of inclusions containing transformation strains or eigenstrains is explicitly derived, the corresponding inhomogeneity problem leads to integrals equations which do not appear to have closed-form solutions. To that end, Eshelby's equivalent inclusion method is extended to the present framework in form of a perturbation series that then can be used to approximate the elastic state of inhomogeneities. The approximate scheme for inhomogeneities also serves as the basis for establishing expressions for the effective properties of composites in second gradient elasticity with couple stresses. The present work is expected to find application towards nano-inclusions and certain types of composites in addition to being the basis for subsequent non-linear homogenization schemes.

1. Introduction and background

One expects a departure from classical continuum mechanics (which is intrinsically size-independent) at the nanoscale. The so-called "size-dependency" and scaling of mechanical phenomena has acquired considerable attention in recent times under various contexts e.g. thin films, quantum dots, plasticity, nanowires and nanotubes, composites among others. The physical mechanisms that appear to cause a deviation from the size-independence of continuum mechanics (whether it is purely linear elastic behavior or non-linear

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plastic behavior) are wide and varied. Insofar as elasticity is concerned (which is the scope of the present work) chiefly three physical reasons may be attributed to a size-dependent elastic behavior. A good example to keep in mind throughout this discussion is the classical problem of a hole in an infinite plate under stress. Stress concentration (in line with the supposed size-independence of classical elasticity) does not vary with the hole-size. Even more generally, as evident from Eshelby's solution of an embedded inclusion (1957, 1959), the elastic state depends solely on the inclusion shape and not on its size.

The first physical mechanism that is responsible for size-effects is the increasing role of surface energies the effect of which becomes appreciable at the nanoscale due to a large surface-to-volume ratio. Several researchers have addressed this issue in various contexts. See for example, the work of Cammarata (1994) who addresses thin films, Miller and Shenoy (2000) who discuss beams and plates, and finally Sharma et al. (2003), Sharma and Ganti (2004, in press), and Sharma and Wheeler (submitted for publication) who focus on the impact of surface energies on nanoinclusions. The second physical cause to which size-dependent elastic behavior can be attributed is the presence of internal motions within a non-primitive lattice (for crystalline materials) or other internal motions above and beyond the classical displacement degrees of freedom (say, for example, in liquid crystals, polymers, and granular materials). This physical phenomenon can be mimicked via the director field theories. A prominent example of such classes of theories is the micromorphic elasticity theory pioneered by and recently reviewed extensively by Eringen (1999). Regarding work in inclusion problem, see the work of Cheng and He (1995, 1997), and Sharma and Dasgupta (2002) (see Fig. 1)

Finally, one may claim that long range interactions between atoms become appreciable at small length scales. Thus, the "long wavelength" assumption of classical elasticity breaks down when one approaches the length scale comparable to the discrete structure of matter (e.g. lattice parameter). At such small length scales where the discrete nature of matter plays a more direct role, additional physical phenomenon not readily included in classical continuum mechanics framework become important. As one would expect, several phenomena at the level of a few lattice spacing are inadequately captured by classical elasticity and researchers often see enriched continuum theories like non-local elasticity (of which strain gradient elasticity is an example) as a replacement for atomistic simulations (or alternatively a bridge between atomistic and conventional continuum mechanics). For example, the ubiquitous singularities ahead of crack tips and dislocation cores (as predicted by classical mechanics) are indeed a break down of traditional elasticity at short wavelengths (Eringen, 2002).

Briefly, in non-local elasticity, the algebraic constitutive equations are replaced by integral equations whereby the stress or strain at a point depends not only on the strain or stress at that point but also on all neighboring points in the material. Under certain conditions, an approximation to the true non-local material behavior can be mimicked by the so-called strain gradient elasticity where strain gradients with suitable coupling constants are added to the classical elastic Lagrangian. Pioneering work in this direction can be

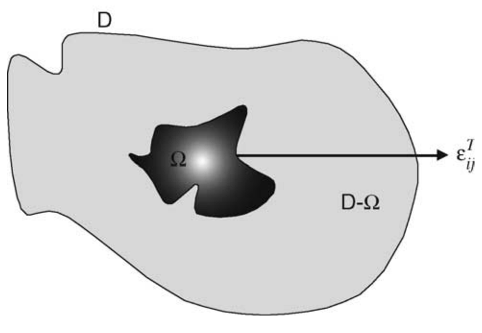

Fig. 1. Schematic of inclusion problem.

traced to Toupin (1962), Mindlin (1964, 1965), Krumhansl (1968), and Eringen and Edelen (1972). Other works of note that have appeared are due to Kunin (1982, 1983), Ru and Aifantis (1993), Gutkin and Aifantis (1996), Aifantis (2003), and Gutkin (2000). In particular, the latter two papers provide comprehensive review of strain gradient theory in the context of defects. The monograph by Eringen (1999) also provides a summary of much of the work done in non-local elasticity. Here we have neglected to mention several outstanding works on non-local theories in the context of plasticity (see for example, Fleck and Hutchinson, 2001 and the references therein—which though briefly also allude to strain gradient elastic behavior). In the present work, as will be discussed shortly, we will follow a strain gradient theory (which incorporates couple stresses) as proposed by Kleinert (1989). The latter is a simplified version of Mindlin's (1965) 2nd gradient theory whereby only up to first gradients of strain are incorporated (which is in line with what most other strain gradient theories do in any case). The complete 2nd gradient theory of Mindlin is, unfortunately, intractable at times for even some simple one-dimensinonal problems although it does have the attractive feature of incorporating directly the first physical mechanism behind elastic size dependency that we mentioned earlier i.e. surface energies. Considering the prevalence of various strain gradient theories in the literature, we provide in Appendix A a comparison of various strain gradient elasticity theories.

In the present work, we tackle the following problems:

(i) The size-dependent elastic state of an *inclusion* [1] within the formalism of a strain gradient elasticity theory with couple stresses.
(ii) The *average* size-dependent elastic state of an *inhomogeneity* [1] within the formalism of a strain gradient elasticity theory with couple stresses.
(iii) The effective size-dependent elastic properties of a heterogeneous solid (composite) within the formalism of a strain gradient elasticity theory with couple stresses.

We hardly need to emphasize the importance of the aforementioned problems. Since the pioneering work of Eshelby (1957) who first solved the three problems indicated above, innumerous investigations have appeared that have extended Eshelby's original work and this area continues to attract significant attention from present day researchers. We do not attempt a detailed review of this body of and simply cite several recent works that have provided comprehensive review of this subject e.g. see, Mura (1987), Nemat-Nasser and Hori (1999), Markov and Preziosi (2000), Weng et al. (1990), Bilby et al. (1984), Mura et al. (1996), Torquato (2001), and Milton (2001) for an in-breadth and in-depth discussion of inclusion and homogenization problems.

Eshelby's (1957) original work related the actual strain ε_{ij} developed in an inclusion (located in an infinite matrix) to the eigenstrain or transformation strain ε^T_{ij} via what is now widely known as the Eshelby's tensor (**S**).

$$\varepsilon_{ij} = S_{ijkl}\varepsilon^T_{kl} \tag{1}$$

In various physical problems, the eigenstrain can represent thermal mismatch, lattice mismatch, phase transformation, and other inelastic transformations. Eshelby's solution is of great versatility and has been employed to address a wide range of physical problems in materials science, mechanics, and solid state physics. Eshelby's tensor, in accordance with the intrinsic size-independence of classical elasticity depends solely on material properties and shape (but not on the size). For example, in the case of isotropic spherical inclusions, Eshelby's tensor can be written explicitly as (Eshelby, 1957; Mura, 1987; Cheng and He, 1995):

[1] As per Mura's nomenclature (1987), an inclusion is bounded region within a material with the same material properties as that of the surrounding material but containing a stress-free transformation strain or eigenstrain. In contrast, an inhomogeneity in a bounded region within a material with differing material properties and may (or may not) contain an eigenstrain. Eshelby (1957) showed that an inhomogeneity under loading can be simulated via an equivalent inclusion containing a fictitious eigenstrain.

$$S_{ilmn} = \frac{3\lambda + 8\mu}{15(\lambda + 2\mu)} \delta_{im}\delta_{ln} + \frac{3\lambda + 8\mu}{15(\lambda + 2\mu)} \delta_{in}\delta_{lm} + \frac{3\lambda - 2\mu}{15(\lambda + 2\mu)} \delta_{il}\delta_{mn} \tag{2}$$

where λ and μ are the Lame constants.

Further, as well known, for uniform eigenstrains, the stresses and strains within the inclusion are also uniform as long as the inclusion shape belongs to the ellipsoid family (which includes, of course, the spherical shape). This fact greatly simplifies the solution to the inhomogeneity problem. In the latter case, Eshelby proposed that an inhomogeneity under applied load can be mimicked or replaced by an inclusion containing a fictitious eigenstrain which can then be determined through the following equivalency condition applied within the volume of the inhomogeneity:

$$C^{\mathrm{I}}_{ijkl}\left(\varepsilon_{kl} + \varepsilon^{\infty}_{kl}\right) = C^{\mathrm{M}}_{ijkl}\left(\varepsilon_{kl} + \varepsilon^{\infty}_{kl} - \varepsilon^{\mathrm{f}}_{kl}\right) \tag{3}$$

Here superscripts "I" and "M" indicate "inhomogeneity" and "matrix" respectively. In this paper, we will also use superscripts "2" for inhomogeneity and "1" for matrix. ε^{f} is the fictitious eigenstrain designed to simulate the perturbation in applied strain due to the modulus mismatch between the inhomogeneity and the matrix while ε^{∞} is the applied strain. Clearly, Eq. (3) is an algebraic one only for ellipsoidal inclusions where Eshelby's tensor is uniform and known explicitly. For other shapes (say polyhedral), Eshelby's tensor is non-uniform and one must then solve an integral equation for the non-uniform fictitious eigenstrain and eventually the elastic state (see for example, Nozaki and Taya, 2001). Needless to say, only approximate solutions are possible in such cases.

In so far as Eshelby's inclusion problem within the context of strain gradient elasticity is concerned, a comprehensive solution appears to be missing in the literature. Various works have addressed bits and pieces of this problem under specific contexts; for example Reid and Gooding (1992) discuss a two-dimensional inclusion problem where the eigenstrain is restricted to a pure dilatation. Wang (1990), too considers a rather specific antiplane eigenstrain problem in two-dimensional non-local elastostatics. In the present work, the complete three-dimensional (strain gradient) Eshelby tensor for spherical inclusion is evaluated in closed form. While for the inclusion problem, our solution is exact, a corresponding solution for inhomogeneity does not appear to be possible. Hence, we address the analytically insoluble integral equations of the inhomogeneity problem using a perturbation expansion type approach. The latter is then used to formulate the effective size-dependent properties of composites. While the scope of the present work is restricted to elasticity, as shown by Ponte Castañeda (1992), using the concept of linear comparison material, our work may provide the basis for future non-linear homogenization schemes.

The outline of our paper is as follows. In Section 2, the general inclusion problem in strain gradient elasticity (incorporating couple stresses) is formulated. We specialize our solution to the spherical shape in Section 3, where closed form expressions are given. In particular, the case of dilatational eigenstrain leads to exceedingly simple expressions and results. In Section 4, perturbation method is employed to solve the inhomogeneity problem. In Section 5, we derive (and present results for) the effective elastic properties of a strain-gradient composite material containing small sized inclusions. We finally conclude with a summary and our major conclusion in Section 6.

2. The general size-dependent inclusion problem in strain gradient elasticity with couple stresses

Consider an arbitrarily shaped inclusion with a prescribed stress-free eigenstrain in its domain (Ω) located in an infinite amount of material (D). As a departure from classical elasticity, we now postulate that either the size-scale of the inclusion is "small" or high field gradients are suspected.

Within the assumption of linear isotropic classical elasticity, the strain energy function is quadratic in strains:

$$W(x) = \mu u_{i,j}u_{i,j} + \frac{1}{2}\lambda u_{l,l}^2 \tag{4}$$

Here, $u_{i,j}$ and $\partial_j u_i$ will be used interchangeably to indicate differentiation with respect to x_j. Note that the anti-symmetric part of the deformation gradient i.e. ω (=asym $\nabla\mathbf{u}$) is absent from Eq. (4) since the quadratic term in ω is not rotationally invariant—a necessary requirement for the energy function in Eq. (4). Large strain gradients in the material however, require that higher order derivatives of rotation also contribute to the elastic energy and this is accomplished via gradients of ω. Indeed, the gradients of ω are admissible since those fields are invariant with respect to the Euclidean group of transformations SO(3)⊳T(3). The general form of the elastic energy involving first gradients of strain and rotation that is invariant to SO(3)⊳T(3) group is (Kleinert, 1989):

$$W(x) = W(\partial_i u_j, \partial_l u_l, \partial_i\partial_l u_l, \partial_i\partial_l u_i) \tag{5}$$

Further discussion is restricted to isotropic strain gradient elasticity. The energy density then takes the form:

$$W(x) = \frac{\mu}{2}(\partial_i u_j)^2 + \frac{\mu+\lambda}{2}(\partial_l u_l)^2 + \frac{2\mu+\lambda}{2}l'^2\partial_i\partial_l u_l\partial_i\partial_j u_j + \frac{\mu l^2}{2}(\partial_l^2 u_i\partial_l^2 u_i - \partial_i\partial_l u_l\partial_i\partial_j u_j) \tag{6}$$

Two new coupling constants (in addition to the Lame' parameters) now appear namely l' and l. Both have units of length. From the energy density expression in Eq. (6), via appeal to the Euler–Lagrange equations, the Navier-like governing static equation can be derived as well as the response quantities (i.e. "stresses"). The balance laws can then be written as

$$\begin{aligned} &\partial_i m_{ij} = -\varepsilon_{jkl}\sigma_{kl}^a \\ &\sigma_{ij} = \sigma_{ij}^a + \sigma_{ij}^s + \delta_{ij}\partial_k\tau_k \\ &\sigma_{ji,j} = 0 \end{aligned} \tag{7a–c}$$

Here $\boldsymbol{\sigma}^{\mathbf{s}}$ is the symmetric part of the stress tensor (and thus coincides with the classical definition of stress typically adopted). The remaining quantities, $\boldsymbol{\tau}$ and $\mathbf{m}$, denote moment-stress like quantities respectively characterizing resistance to strain gradients and rotational gradients. The physical stress tensor ($\boldsymbol{\sigma}$) is then defined from these and yields (superficially) a balance law (7c) that appears superficially the same as in classical elasticity (Kleinert, 1989). ε_{ijk} here stands for the permutation symbol.

For materials that obey linear isotropic constitutive relations and subject to the usual symmetry and invariance constraints, we can define the following constitutive relations (Kleinert, 1989):

$$\begin{aligned} &\sigma_{ij}^s = 2\mu u_{i,j} + \lambda\delta_{ij}u_{l,l} \\ &\tau_i = (2\mu+\lambda)l'^2\partial_i u_{l,l} \\ &m_{ij} = 4\mu l^2\partial_i\omega_j \end{aligned} \tag{8a–c}$$

For a comparison of the strain gradient model that we employ in the present work and others that are prevalent in the literature, please see Appendix A.

We now tackle the inclusion problem. Noting that the transformation strain is only non-zero within the inclusion domain, we can write the bulk-constitutive law for the inclusion-matrix as follows:

$$\begin{aligned} &m_{ij} = 4\mu l^2\partial_i(\omega_j - \omega_j^* H) \\ &\varepsilon_{ijk}\sigma_{jk}^a = -\partial_j m_{ij} \\ &\sigma_{ij} = \sigma_{ij}^a + 2\mu(u_{i,j} - u_{i,j}^* H) + \lambda\delta_{ij}(u_{l,l} - u_{l,l}^* H) + \delta_{ij}(2\mu+\lambda)l'^2(\nabla^2(u_{l,l} - u_{l,l}^* H)) \end{aligned} \tag{9a–c}$$

Here "H" is Heaviside equation, defined as $H(x)=\begin{cases}1 & x\in\Omega\\ 0 & x\notin\Omega\end{cases}$. Using the equilibrium equations and the constitutive relations, we can obtain the Navier-like governing equation for this problem i.e.

$$-\mu\partial^2 u_i-(\mu+\lambda)\partial_i\partial_l u_l+(2\mu+\lambda)l'^2\partial^2\partial_i\partial_l u_l+\mu l^2(\partial^4 u_i-\partial^2\partial_i\partial_l u_l)=\partial_k(P^T_{ik}H) \tag{10}$$

Here the symbol $\mathbf{P^T}$ stands for the eigenstress corresponding to the eigenstrain, i.e. $C_{jlmn}\varepsilon^*_{mn}$.

Clearly, identical to the classical case (Mura, 1987), the derivative of the eigenstrain acts as a body force. Noting that the derivatives of the Heaviside function defined over the inclusion volume act as delta functions across the inclusion surface, the displacement vector can be obtained using the Green's function (of Eq. (12)) as

$$u_i(x)=\int_S P^T_{jl}G_{ij}(x-x')\,\mathrm{d}S_l(x')=-\int_V C_{jlmn}\varepsilon^*_{mn}(x')G_{ij,l}(x-x')\,\mathrm{d}V(x') \tag{11}$$

Here Gauss theorem has been used to convert the surface integral into a volume integral. Fortunately, strain gradient Green's function has already been derived by Kleinert (1989) and is explicitly given by

$$G_{ij}(R)=\frac{1}{4\pi\mu R}\left(1-e^{-\frac{R}{l}}\right)\delta_{ij}-\frac{1}{4\pi\mu}\partial_i\partial_j\left(\frac{R}{2}+l^2\frac{1}{R}\left(1-e^{-\frac{R}{l}}\right)\right)+\frac{1}{4\pi 2\mu+\lambda}\partial_i\partial_j\left(\frac{R}{2}+l'^2\frac{1}{R}\left(1-e^{-\frac{R}{l'}}\right)\right) \tag{12}$$

where $R=|x-x'|$.

Mere substitution of Eq. (14) into (13) results in:

$$\begin{aligned}u_i(x)&=\frac{1}{4\pi}\int \mathrm{d}S_k(x')\left[\frac{1}{\mu R}\left(1-e^{-\frac{R}{l}}\right)P^T_{ik}-\frac{P^T_{jk}}{\mu}\partial_i\partial_j\left(\frac{R}{2}+l^2\frac{1}{R}\left(1-e^{-\frac{R}{l}}\right)\right)\right.\\&\qquad\left.+\frac{P^T_{jk}}{2\mu+\lambda}\partial_i\partial_j\left(\frac{R}{2}+l'^2\frac{1}{R}\left(1-e^{-\frac{R}{l'}}\right)\right)\right]\\&=-\frac{1}{\mu}(\phi_{,k}-M_{,k})P^T_{ik}+\frac{P^T_{jk}}{\mu}\partial_i\partial_j\left(\frac{\psi_{,k}}{2}+l^2(\phi_{,k}-M_{,k})\right)-\frac{P^T_{jk}}{2\mu+\lambda}\partial_i\partial_j\left(\frac{\psi_{,k}}{2}+l'^2(\phi_{,k}-M'_{,k})\right)\\&=-\frac{1}{\mu}(\phi_{,k}-M_{,k})P^T_{ik}+\frac{P^T_{jk}}{\mu}\left(\frac{\psi_{,ijk}}{2}+l^2\phi_{,ijk}-l^2M_{,ijk}\right)-\frac{P^T_{jk}}{2\mu+\lambda}\left(\frac{\psi_{,ijk}}{2}+l'^2\phi_{,ijk}-l'^2M'_{,ijk}\right)\end{aligned} \tag{13}$$

Here we have made explicit the use of Gauss's theorem to convert the surface integrals to volume integrals and the displacement field has been cast in terms of certain potentials defined below:

$$\begin{aligned}\psi(x)&=\frac{1}{4\pi}\int_\Omega R\,\mathrm{d}x'\\ \phi(x)&=\frac{1}{4\pi}\int_\Omega \mathrm{d}x'\\ M(x,l)&=\frac{1}{4\pi}\int_\Omega \frac{e^{-\frac{R}{l}}}{R}\,\mathrm{d}x'\\ M'(x,l')&=\frac{1}{4\pi}\int_\Omega \frac{e^{-\frac{R}{l'}}}{R}\,\mathrm{d}x'\end{aligned} \tag{14}$$

$\psi(x)$ is the harmonic potential, $\phi(x)$ is the biharmonic potential while $M(x,l)$ and $M'(x,l')$ are the Yukawa potentials with different coefficients l and l' (i.e. the characteristic length scales in gradient elasticity theory).

The first two potentials are well known in classical potential theory (Kellogg, 1953) and the inclusion literature (see, Mura, 1987). The Yukawa potential is relatively less known and occurs in the study of non-Newtonian gravitation (e.g. Gibbons and Whiting, 1981). Recently, it has been employed by Cheng and He (1995, 1997) in their study of inclusions in micropolar elasticity.

For convenience we can write displacement in two parts: the classical one (already detailed by Eshelby (1957) for ellipsoidal inclusions in terms of the harmonic and biharmonic potentials) and the second size-dependent part which arises due to strain gradient effects. From Eq. (15), the strain gradient part can be written as

$$u_i^{\mathrm{GR}} = \left[\frac{M_{,k}}{\mu}\delta_{ij} + \frac{l^2}{\mu}(\phi_{,ijk} - M_{,ijk}) - \frac{l'^2}{2\mu+\lambda}\left(\phi_{,ijk} - M'_{,ijk}\right)\right]P^T_{jk} \tag{15}$$

The symmetric part of the strain, $\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$, is then:

$$\varepsilon_{il}^{\mathrm{GR}} = \left[\frac{1}{2\mu}(M_{,kl}\delta_{ij} + M_{,ki}\delta_{lj}) + \left(\frac{l^2}{\mu} - \frac{l'^2}{2\mu+\lambda}\right)\phi_{,ijkl} - \frac{l^2}{\mu}M_{,ijkl} + \frac{l'^2}{2\mu+\lambda}M'_{,ijkl}\right]P^T_{jk} \tag{16}$$

Using Eshelby's convention, we can then define the complete strain gradient Eshelby tensor to be:

$$\begin{aligned} S_{ijkl} &= S^0_{ijkl} + S^{\mathrm{GR}}_{ijkl} \\ &= S^0_{ijkl} + \frac{1}{2}\left[M_{,lj}\delta_{ik} + M_{,li}\delta_{jk} + M_{,kj}\delta_{il} + M_{,ki}\delta_{jl}\right] \\ &\quad + 2\mu\left[\left(\frac{l^2}{\mu} - \frac{l'^2}{2\mu+\lambda}\right)\phi_{,ijkl} - \frac{l^2}{\mu}M_{,ijkl} + \frac{l'^2}{2\mu+\lambda}M'_{,ijkl}\right] \\ &\quad + \left\{\frac{\lambda}{\mu}M_{,ij} + \lambda\left[\left(\frac{l^2}{\mu} - \frac{l'^2}{2\mu+\lambda}\right)\phi_{,ijmm} - \frac{l^2}{\mu}M_{,ijmm} + \frac{l'^2}{2\mu+\lambda}M'_{,ijmm}\right]\right\}\delta_{kl} \end{aligned} \tag{17}$$

where $\mathbf{S^0}$ is the classical Eshelby's tensor (known for various inclusion shapes—see Mura, 1987) and $\mathbf{S}^{\mathrm{GR}}$ is the gradient part.

Finally, the dilatation can be expressed as

$$\mathrm{tr}(\varepsilon_{il}^{\mathrm{GR}}) = \left[\frac{1}{\mu}M_{,jk} + \left(\frac{l^2}{\mu} - \frac{l'^2}{2\mu+\lambda}\right)\phi_{,jkmm} - \frac{l^2}{\mu}M_{,jkmm} + \frac{l'^2}{2\mu+\lambda}M'_{,jkmm}\right]P^T_{jk} \tag{18}$$

The present formulation was for arbitrary shaped inclusions and we specialize to the spherical shape in the next section to obtain explicit expressions.

3. Spherical inclusions: closed form expressions

Assume a spherical inclusion of radius "a" embedded in an infinite matrix (see Fig. 2).

Our results in the previous section were cast in terms of three potentials, the harmonic, biharmonic and the so-called Yukawa potential. These can be written in closed form for the spherical shape (see, Mura, 1987; Kellogg, 1953; Cheng and He, 1995; Gibbons and Whiting, 1981):

$$\psi(x) = \begin{cases} -\dfrac{1}{60}(R^4 - 10a^2R^2 - 15a^4) & R \in \Omega \\ \frac{a^3}{15}\left(5R + \frac{a^2}{R}\right) & R \notin \Omega \end{cases} \tag{19}$$

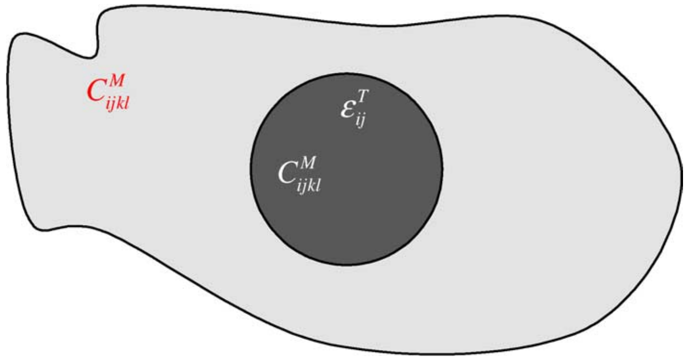

Fig. 2. Spherical inclusion problem schematic.

$$\phi(x) = \begin{cases} -\dfrac{1}{6}(R^2 - 3a^2) & R \in \Omega \\ \dfrac{a^3}{3R^2} & R \notin \Omega \end{cases} \tag{20}$$

$$M(x,k) = \begin{cases} k^2 - k^2(k+a)e^{-a/k}\dfrac{sh(R/k)}{R} & R \in \Omega \\ k^2\left(ach\frac{a}{k} - ksh\frac{a}{k}\right)\dfrac{e^{-R/k}}{R} & R \notin \Omega \end{cases} \tag{21}$$

Using, Eq. (19) and (21)–(23), the strain gradient Eshelby's tensor for the spherical shape can be made explicit. Although we have obtained the complete Eshelby's tensor for gradient elasticity and can thus be employed for arbitrary eigenstrains Eq. (19), exceedingly simple expressions for the dilatation can be derived; which is of frequent interest in various physical problems involving for example, thermal expansion, lattice mismatch, phase transformations etc.

A dilatational eigenstrain implies a dilatational eigenstress $P_{ij}^T = P^T\delta_{ij}$. In this particular simple case, the eigenstrain and eigenstress are related by $P^T = (3\lambda + 2\mu)\varepsilon^T$. Some algebra and manipulations result in following relation for the dilatation:

$$\mathrm{tr}(\varepsilon^{\mathrm{GR}}) = \begin{cases} -\dfrac{P^T}{2\mu+\lambda}(l'+a)e^{-\frac{a}{l'}}\dfrac{sh\frac{R}{l'}}{R} & R \in \Omega \\ \dfrac{P^T}{2\mu+\lambda}\left(ach\dfrac{a}{l'} - l'sh\dfrac{a}{l'}\right)\dfrac{e^{-\frac{R}{l'}}}{R} & R \notin \Omega \end{cases} \tag{22}$$

Adding to it the dilatation from the well-known classical part, we obtain the total trace of the strain as

$$\mathrm{tr}(\varepsilon) = \begin{cases} \dfrac{\varepsilon^T(3\lambda+2\mu)}{\lambda+2\mu}\left[1 - (l'+a)e^{-\frac{a}{l'}}\dfrac{1}{l'}\dfrac{sh\frac{x}{l'}}{\frac{x}{l'}}\right] \\ \dfrac{\varepsilon^T(3\lambda+2\mu)}{\lambda+2\mu}\left(ach\dfrac{a}{l'} - l'sh\dfrac{a}{l'}\right)\dfrac{1}{l'}\dfrac{e^{-\frac{x}{l'}}}{\frac{x}{l'}} \end{cases} \tag{23}$$

The suitably normalized dilatational strain results are plotted in Fig. 3 as a function of position and various inclusion sizes. All the results in the present work are plotted parametrically in terms of the unknown strain gradient constants. Please see Appendix B for more discussion on determination of the actual strain gradient properties. The location $x/a = 1$ indicates the boundary of the spherical inclusion. The size effect of

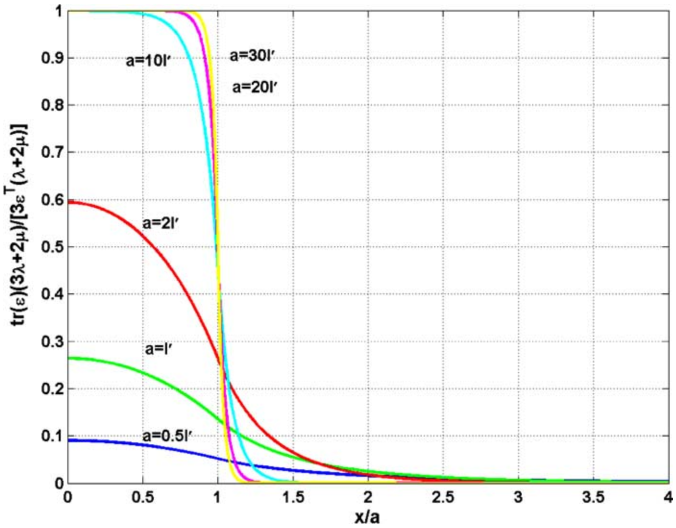

Fig. 3. Strain dilatation as a function of position and inclusion size.

the strain gradient solution is clearly obvious. We note that, unlike the classical solution, the strain gradient solution is inhomogeneous within the inclusion and asymptotically, our results converge to that of classical elasticity (Eshelby, 1957) for large inclusion sizes. As well known, the classical result predicts zero dilatation outside the inclusion for a dilatational eigenstrain. A further point to be noted is that our results (for the dilatation) depend solely on l' and not on l. This is due to the fact that the latter is physically associated with couple stresses or gradients of the rotation vector and (as well known), for isotropic materials, they vanish for purely centrosymmetric loading. The parameter l' is associated with purely gradient effects of

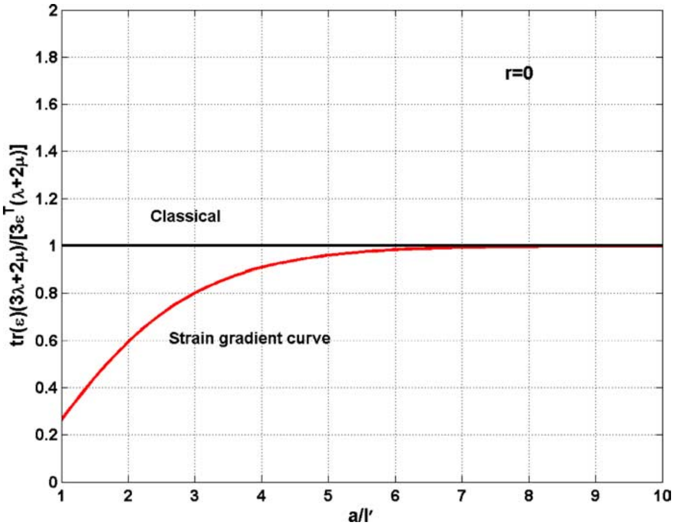

Fig. 4. Dilatational strain as a function of size for fixed position (r = 0).

the symmetric part of the strain and hence persists even in the highly symmetric dilatation case. In other words, the conventional centrosymmetric isotropic micropolar elasticity theory would not differ from classical elasticity for the purely dilatational problem.

To emphasize on the size-dependency of our solution we also plot the dilatation as a function of size (for a fixed position, i.e. $x = 0$) in Fig. 4. We observe that while for large inclusion size, roughly $>a = 7l'$, the strain gradient solution is indistinguishable from the classical one, the dilatation decreases significantly below this threshold.

4. The inhomogeneity problem in strain gradient elasiticty with couple stresses

The inhomogeneity problem is considerably more challenging and as alluded to earlier, unlike for the inclusion problem, an exact general solution does not appear to be possible. Imagine that a far-field strain, $\varepsilon_{ij}^{\infty}$ is applied to the inhomogeneity matrix system. We start of by writing the position dependent elastic modulus in the following fashion:

$$C_{ijkl} = C^1_{ijkl} + \Delta C_{ijkl} H(x) \tag{24}$$

Here, ΔC_{ijkl} is the difference between the elastic stiffness tensors of the matrix and the inhomogeneity: $\Delta C = C^2_{ijkl} - C^1_{ijkl}$. Once again, from the equilibrium equations, $\nabla \cdot \sigma = 0$, we obtain:

$$\partial_j \left(C^1_{ijkl} u_{k,l} \right) + \partial_j \left(H(x) \Delta C_{ijkl} u_{k,l} \right) = 0 \tag{25}$$

This allows us to write (using approach outlined by Markov, 1979), the integral equations for the strain in the inhomogeneity:

$$\begin{aligned} \varepsilon_{ij}(x) &= \varepsilon_{ij}^{\infty}(x) + \int_{\Omega} \mathbb{Q}_{iklj}(x - x') \Delta C_{klmn} \varepsilon_{mn}(x') \, \mathrm{d}^3 x' \\ \mathbb{Q}_{iklj}(x) &= \frac{1}{2} \left(G_{ik,lj}(x) + G_{jk,li}(x) \right) \end{aligned} \tag{26}$$

These integral equations, while not soluble exactly, may be tackled by a perturbation type approach. In such a case, the first approximation to the actual strain can be considered to be the "average" uniform strain. We now proceed to evaluate this average strain from the point of view of later application to the effective properties of composites.

We perform a volumetric averaging process over both sides of Eq. (28):

$$\langle \varepsilon_{ij}(x) \rangle = \varepsilon_{ij}^{\infty}(x) + \left\langle \int_{\Omega} \mathbb{Q}_{iklj}(x - x') \Delta C_{klmn} \varepsilon_{mn}(x') \, \mathrm{d}^3 x' \right\rangle \tag{27}$$

As a first approximation, we assume that:

$$\left\langle \int_{\Omega} \mathbb{Q}_{iklj}(x - x') \Delta C_{klmn} \varepsilon_{mn}(x') \, \mathrm{d}x' \right\rangle = \left\langle \int_{\Omega} \mathbb{Q}_{iklj}(x - x') \Delta C_{klmn} \, \mathrm{d}x' \right\rangle \langle \varepsilon_{mn}(x') \rangle$$

It can be shown (see Markov, 1979, who uses this in the context of micropolar elasticity where also strains are non-uniform) that this approximation is tantamount to adopting the first term in a perturbation series expansion involving the difference in the moduli of the inhomogeneity-matrix. Implicitly or explicitly, other researchers have also employed such an approximation where strain states are inhomogeneous e.g. Nozaki and Taya (2001) in the context of polyhedral inhomogeneities and Sharma and Dasgupta (2002) in the case of micropolar inhomogeneities.

Thus we can relate/approximate the average elastic strain inside inhomogeneity to the fictitious eigenstrain in terms of the "average strain gradient Eshelby tensor": $\langle \varepsilon_{ij} \rangle = \langle S_{ijkl}\varepsilon^f_{kl} \rangle \sim \langle S_{ijkl} \rangle \langle \varepsilon^f_{kl} \rangle$.

The average modified Eshelby tensor in second strain gradient elasticity is:

$$\begin{aligned}
\langle S_{ilmn}\rangle &= \langle S_{ilmn}\rangle^0 + \langle S_{ilmn}\rangle^{\mathrm{GR}} \\
\langle S_{ilmn}\rangle^0 &= \frac{3\lambda_1+8\mu_1}{15(\lambda_1+2\mu_1)}\delta_{im}\delta_{ln} + \frac{3\lambda_1+8\mu_1}{15(\lambda_1+2\mu_1)}\delta_{in}\delta_{lm} + \frac{3\lambda_1-2\mu_1}{15(\lambda_1+2\mu_1)}\delta_{il}\delta_{mn} \\
\langle S_{ilmn}\rangle^{\mathrm{GR}} &= \left\langle \left[\frac{1}{2\mu_1}\left(M_{,kl}\delta_{ij}+M_{,ki}\delta_{lj}\right) - \frac{l^2}{\mu_1}M_{,ijkl} + \frac{l'^2}{2\mu_1+\lambda_1}M'_{,ijkl}\right]C_{jkmn}\right\rangle \\
&= \left\{-\frac{1}{6\mu_1}f(\delta_l)\left[\delta_{kl}\delta_{ij}+\delta_{ki}\delta_{lj}\right] + \left[\frac{1}{15\mu_1}f(\delta_l) - \frac{1}{15(2\mu_1+\lambda_1)}f\left(\delta'_l\right)\right]H_{ijkl}\right\}C_{jkmn} \\
&= -\left\{\left[\frac{1}{5\mu_1}f(\delta_l) + \frac{2}{15(2\mu_1+\lambda_1)}f\left(\delta'_l\right)\right]C_{ilmn} + \left[-\frac{1}{15\mu_1}f(\delta_l) + \frac{1}{15(2\mu_1+\lambda_1)}f\left(\delta'_l\right)\right]C_{kkmn}\delta_{il}\right\}
\end{aligned} \tag{28}$$

Following Markov (1979) who encountered similar terms, we set

$$A = \frac{1}{5\mu_1}f(\delta_l) + \frac{2}{15(2\mu_1+\lambda_1)}f\left(\delta'_l\right) \quad \text{and} \quad B = -\frac{1}{15\mu_1}f(\delta_l) + \frac{1}{15(2\mu_1+\lambda_1)}f\left(\delta'_l\right)$$

The function $f(\delta_l) = 3\frac{1+\delta_l}{\delta_l^3}e^{-\delta_l}(\delta_l ch\delta_l - sh\delta_l)$ with $\delta_l = a/l$

Hence, more compactly,

$$\langle S_{ilmn}\rangle = g_1\delta_{im}\delta_{ln} + g_1\delta_{in}\delta_{lm} + g_2\delta_{il}\delta_{mn} \tag{29}$$

where

$$g_1 = -\mu_1 A + \frac{3\lambda_1+8\mu_1}{15(\lambda_1+2\mu_1)} \quad \text{and} \quad g_2 = -\lambda_1 A - B(2\mu_1+3\lambda_1) + \frac{3\lambda_1-2\mu_1}{15(\lambda_1+2\mu_1)}$$

Given our approximation of an "average" uniform eigenstrain in the inhomogeneity, we can now employ Eshelby's equivalent inclusion concept i.e. $C^1_{ijkl}\left(\langle\varepsilon_{kl}\rangle + \langle\varepsilon^\infty_{kl}\rangle - \langle\varepsilon^f_{kl}\rangle\right) = C^2_{ijkl}\left(\langle\varepsilon_{kl}\rangle + \langle\varepsilon^\infty_{kl}\rangle\right)$. In other words:

$$\begin{aligned}
\varepsilon^f_{jk} &= \left(H_1\delta_{ji}\delta_{kl} + H_1\delta_{jl}\delta_{ki} + H_2\delta_{jk}\delta_{il}\right)\varepsilon^\infty_{il} \\
H_1 &= -\frac{\Delta\mu}{2(2g_1\Delta\mu+\mu_1)} \\
H_2 &= \frac{g_2\Delta\mu\Delta K + \frac{\lambda_1\Delta K - K_1\Delta\lambda}{2}}{(2g_1\Delta\mu+\mu_1)(2g_1\Delta K + 3\Delta K g_2 + K_1)}
\end{aligned} \tag{30}$$

The equations developed above allow an approximate evaluation of the average strain within an inhomogeneity located in a strain gradient material that admits couple stresses. For more accurate assessment, yet more terms (i.e. linear, quadratic etc.) in the series expansion must be considered.

5. Effective size-dependent non-local properties of composites

To obtain the size dependent effective properties of composites containing many inhomogeneities, we employ the conventional formalism of concentration factors, see for example, Torquato (2001) and Markov and Preziosi (2000). Concentration factor, A_{ijkl}, connects the far-field strain at infinity to the average elastic strain inside the inhomogeneity (and is sometimes called Wu's tensor).

The total average strain in the inhomogeneity is: $\langle\varepsilon\rangle^{in} = \langle\varepsilon\rangle + \varepsilon^{\infty} = \langle S\varepsilon^{f}\rangle + \varepsilon^{\infty} \sim \langle S\rangle\langle\varepsilon^{f}\rangle + \varepsilon^{\infty}$. Here, **S**, is the complete strain gradient Eshelby tensor derived in this work. Implicit in our approximation (of the inhomogeneity problem) is that terms higher order than linear in difference moduli i.e. $O(\Delta C^2)$ are neglected.

Employing the results of the previous section, we can then compactly write the concentration factor as

$$A_{mnil} = A_1\delta_{mi}\delta_{nl} + A_1\delta_{ml}\delta_{ni} + A_2\delta_{mn}\delta_{il}$$
$$A_1 = \frac{\mu_1}{2}\cdot\frac{1}{(2g_1\Delta\mu + \mu_1)} \quad (31)$$
$$A_2 = \frac{(\lambda_1\Delta K - K_1\Delta\lambda)(2g_1 + 3g_2) - 2g_2\Delta\mu K_1}{2(2g_1\Delta\mu + \mu_1)(2g_1\Delta K + 3g_2\Delta K + k_1)}$$

Setting $\alpha = 2g_1 + 3g_2$ and $\beta = 2g_1$, we write:

$$A_{ijkl} = \frac{K_1}{K_1 + \alpha\Delta K}I'_{ijkl} + \frac{\mu_1}{\mu_1 + \beta\Delta\mu}I''_{ijkl}$$
$$I'_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl} \quad (32)$$
$$I''_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right)$$

Here, K is bulk modulus. Eq. (32) appears identical to the classical results. The sole difference is that coefficients α and β are redefined:

$$\alpha = \underbrace{\frac{3\lambda_1 + 2\mu_1}{3(\lambda_1 + 2\mu_1)}}_{\text{classical}} - \underbrace{(2\mu_1 + 3\lambda_1)(A + 3B)}_{\text{non-classical}}$$
$$\beta = \underbrace{\frac{2(3\lambda_1 + 8\mu_1)}{15(\lambda_1 + 2\mu_1)}}_{\text{classical}} - \underbrace{2\mu_1 A}_{\text{non-classical}} \quad (33)$$

Here A and B are defined in the previous section. Having defined the concentration factors, we can employ any number of homogenization methods (e.g. self-consistent scheme, the effective field or Mori–Tanaka method etc.). Since, for two phase materials with isotropic distribution of spherical particles, effective field or Mori–Tanaka method is reasonable and yet provides analytical expression, we adopt it in the following. In the effective field method (see for example, Markov and Preziosi, 2000), the influence of other (finite concentration of) inhomogeneities is mimicked by an effective uniform field. Once the concentration factor is known (which we have already derived), the effective modulus within the effective field or Mori–Tanaka scheme can be obtained via (Markov and Preziosi, 2000):

$$C^{*}_{ijkl} = C^{1}_{ijkl} + \phi_2 A_{ijmn}\Delta C_{mnst} : (\phi_1 I_{stkl} + \phi_2 A_{stkl})^{-1} \quad (34)$$

C^{*}_{ijkl} is the effective size-dependent stiffness tensor while ϕ_1 and ϕ_2 are the volume fractions of the matrix and inhomogeneities respectively and satisfy: $\phi_1 + \phi_2 = 1$.I_{ijkl} is the fourth order unit tensor which has the form:

$$I_{ijkl} = \frac{1}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})$$

Eq. (35) is exactly the expression of effective property under effective field theory. In appearance it is identical to the classical elasticity solution. The difference of course lies in the redefinition of the concentration factor (Eqs. (31)–(33)).

From Eq. (36) we can generate direct expressions for the effective bulk modulus K^* and the effective shear modulus μ^*:

$$K^* = K_1 + \frac{K_1\phi_2\Delta K}{K_1 + \alpha\phi_1\Delta K}$$
$$\mu^* = \mu_1 + \frac{\mu_1\phi_2\Delta K}{\mu_1 + \beta\phi_1\Delta\mu} \quad (35a\text{-}b)$$

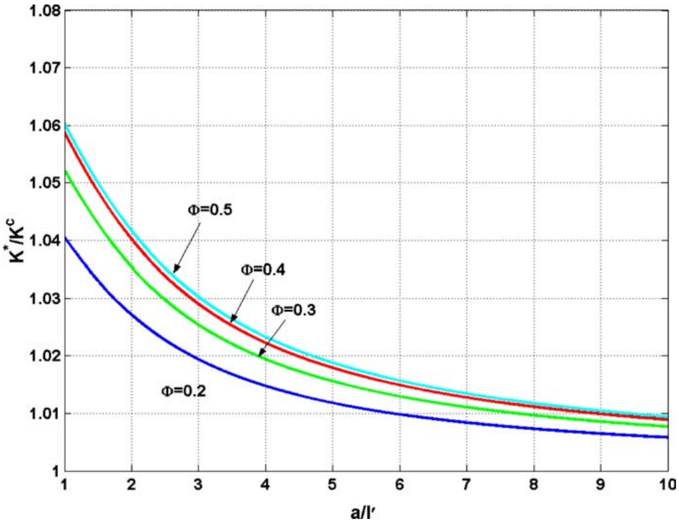

Fig. 5. Effective bulk modulus with $K_2 = 2K_1$ and $\lambda_1 = \mu_1$.

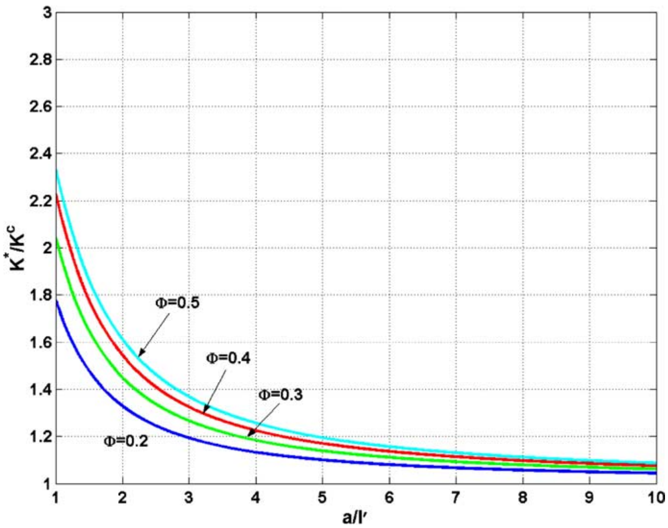

Fig. 6. Effective bulk modulus with $K_2 = 20K_1$ and $\lambda_1 = \mu_1$.

We first present results for the effective bulk modulus in Fig. 5 for various volume fractions and as a function of inclusion size. The following normalization is employed: $\bar{K} = K^*/K^c$ and $\bar{x} = a/l$ where K^c is the effective bulk modulus predicted by classical elasticity. Fig. 5 is plotted for $K_2 = 2K_1$ $\lambda_1 = \mu_1$. As expected, for large inclusion sizes, our solution approaches the classical size-independent one.

In Fig. 5, the departure from classical solution is weak since the difference in properties of the matrix and inclusion is not very large. In Fig. 6 we plot results with the bulk moduli ratio of inhomogeneity to matrix equal to 20. Significant departure from classical results is now seen at small inclusion sizes.

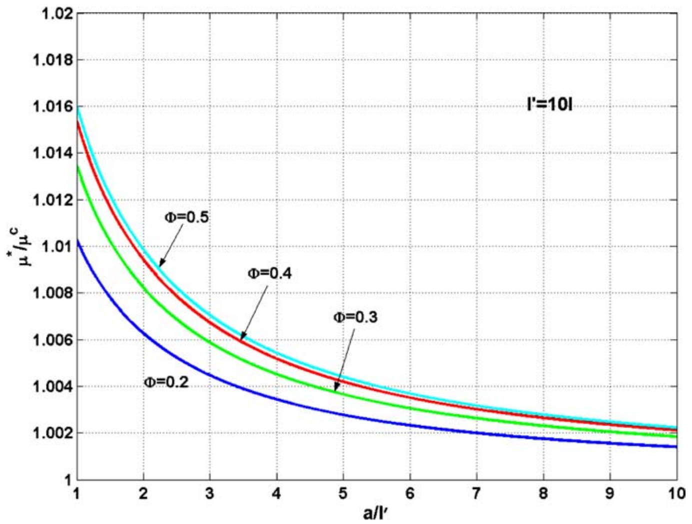

Fig. 7. Effective shear modulus with $\mu_2 = 2\mu_1$, $\lambda_1 = \mu_1$.

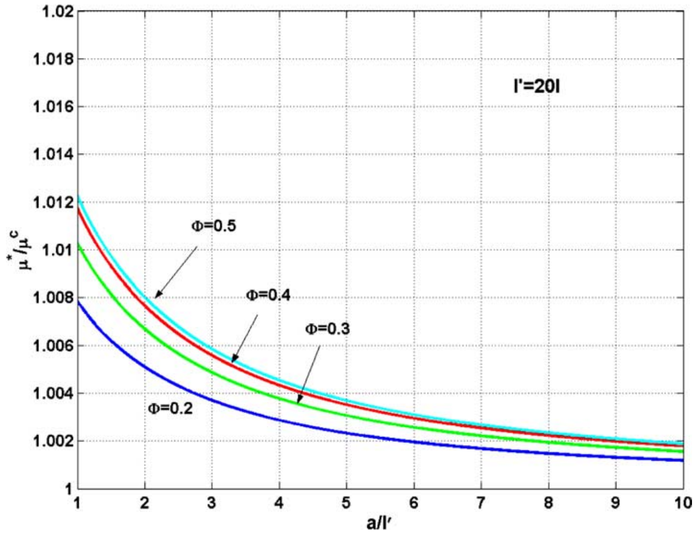

Fig. 8. Effective shear modulus with $\mu_2 = 3\mu_1$, $\lambda_1 = \mu_1$, $l' = 20l$.

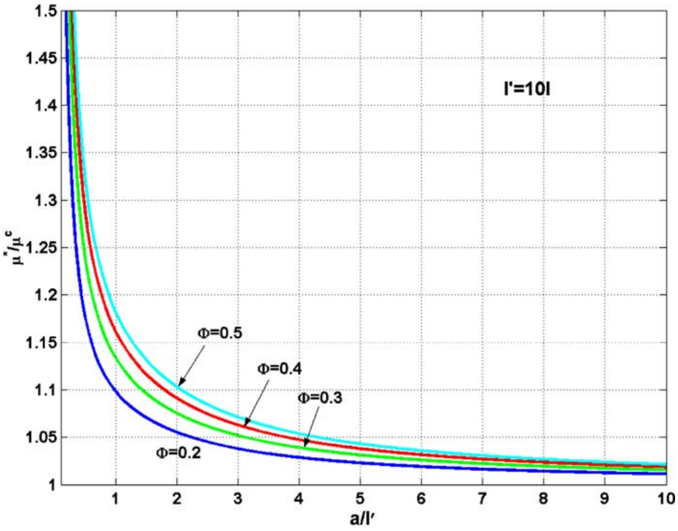

Fig. 9. Effective shear modulus with $\mu_2 = 21\mu_1$, $\lambda_1 = \mu_1$, $l' = 10l$.

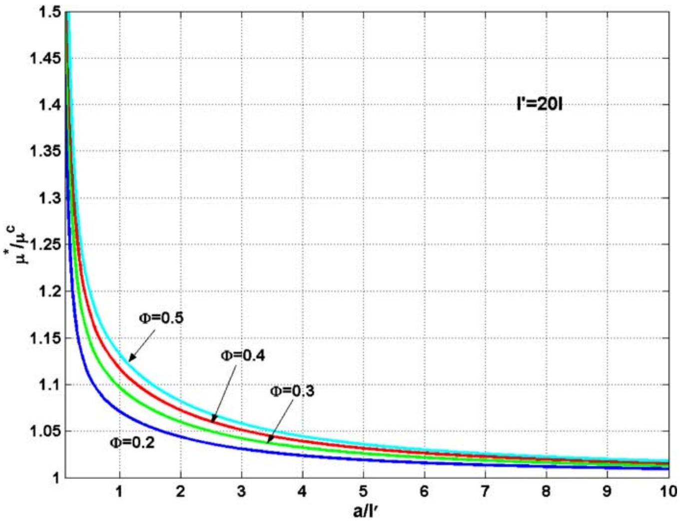

Fig. 10. Effective shear modulus with $\mu_2 = 21\mu_1$, $\lambda_1 = \mu_1$, $l' = 20l$.

Similar plots are shown for the effective shear modulus in Figs. 7–10. The following normalization is employed: $\bar{\mu} = \mu^*/\mu^c$ and $\bar{x} = a/l$. Note that unlike in the case of bulk modulus, the effective shear modulus depends both on l and l'.

6. Summary and conclusions

In summary, we have extended Eshelby's classical approach towards inclusions and inhomogeneities to incorporate the size effect via the concept of strain gradient elasticity. The general form of modified strain gradient Eshelby's tensor for arbitrary shaped inclusions was given in terms of three potentials. Explicit and exact solution was provided for special case of a spherical inclusion. An approximate solution to the inhomogeneity problem was also provided based on which and with an appeal to the effective field theory, explicit expressions for the effective size-dependent bulk and shear modulus of a composites material were derived.

We anticipate several applications and possible improvements of the present work. Clearly, the present work can be employed to reanalyze several classical problems in the nanoregime e.g. phase transformation at small scales, thermal mismatch problem for nanoinclusions, lattice mismatch problem in quantum dots etc. While our solution to the inclusion problem is exact, the inhomogeneity problem was solved with the assumption of a "uniform" strain approximation. This is tantamount to adopting only the first term in a perturbation series expansion in terms of the difference in the elastic moduli of the inclusion-matrix system. Further terms must be incorporated for more accurate results. Use of the present work for constructing effective property solutions for non-linear gradient materials is also relegated to future work.

Appendix A. Comparison of various strain gradient theories

The table below summarizes the various strain gradient theories and their differences/similarities. This compilation is by no means complete but serves to provide a benchmark for the more prevalent ones. Only strain gradient theories are listed (for example, Eringen's non-local integral formulation is not compared).

Model (Reference)	Description	Comparison to other works
Mindlin's 2nd gradient model (1965)	$W = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + a_1\varepsilon_{ijj}\varepsilon_{ikk} + a_2\varepsilon_{iik}\varepsilon_{kjj} + a_3\varepsilon_{iik}\varepsilon_{jjk} + a_4\varepsilon_{ijk}\varepsilon_{ijk} + a_5\varepsilon_{ijk}\varepsilon_{kji} + b_1\varepsilon_{iijj}\varepsilon_{kkll} + b_2\varepsilon_{ijkk}\varepsilon_{ijll} + b_3\varepsilon_{iijk}\varepsilon_{jkll} + b_4\varepsilon_{iijk}\varepsilon_{llkj} + b_5\varepsilon_{iijk}\varepsilon_{lljk} + b_6\varepsilon_{ijkl}\varepsilon_{ijkl} + b_7\varepsilon_{ijkl}\varepsilon_{jkli} + c_1\varepsilon_{ii}\varepsilon_{jjkk} + c_2\varepsilon_{ij}\varepsilon_{ijkk} + c_3\varepsilon_{ij}\varepsilon_{kkij} + b_0\varepsilon_{iijj}$ $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ $\varepsilon_{ijk} = u_{i,jk}$ $\varepsilon_{ijkl} = u_{i,jkl}$	The most generalized strain gradient model that includes also other higher order stresses (beyond couple stresses). This formulation which contains up to 2nd gradients of strains has the advantage to also incorporate surface energies. Most other strain gradient theories can be obtained by appropriate simplification of this work
Kleinert's model (1989—used in the present work)	$W(x) = \frac{\mu}{2}(\partial_i u_j)^2 + \frac{\mu+\lambda}{2}(\partial_l u_l)^2 + \frac{2\mu+\lambda}{2} l'^2 \partial_i\partial_l u_l \partial_i\partial_j u_j + \frac{\mu l^2}{2}(\partial_l^2 u_i \partial_l^2 u_i - \partial_i\partial_l u_l \partial_i\partial_j u_j)$ The appropriate Navier-like equation is: $-\mu\nabla^2 u_i - (\mu+\lambda)\partial_i\partial_l u_l + (2\mu+\lambda)l'^2\nabla^2\partial_i\partial_l u_l + \mu l^2(\nabla^4 u_i - \nabla^2\partial_i\partial_l u_l) = f_i(x)$	This model can be obtained from Mindlin's (1965) formulation by setting: $a_2 = a_4 = a_5 = b_n = c_n = 0$ Additionally, the direct derivation by Kleinert of this model is very appealing

Appendix A (*continued*)

Model (Reference)	Description	Comparison to other works
Koiter's model (1964)	$W = \frac{1}{2}\lambda(\mathrm{tr}\,\varepsilon)^2 + \mu\varepsilon_{ij}\varepsilon_{ij} + l^2(x_{ij}x_{ij} + vx_{ij}x_{ji})$ in above function: $x_{ij} = \frac{1}{2}\varepsilon_{ihk}u_{k,hj}$ Navier-like equation is: $\sigma_{ij,j} - l^2\nabla^2(u_{i,jj} - u_{j,ji}) = 0$	Koiter's model is simplified version of Kleinert's and can be obtained by setting $l' = 0$
Yang et al.'s model (2002)	$W = \frac{1}{2}\lambda(\mathrm{tr}\,\varepsilon)^2 + \mu(\varepsilon_{ij}\varepsilon_{ij} + l^2\chi_{ij}\chi_{ij})$ in the above function: $\chi_{ij} = -\frac{1}{4}(\varepsilon_{ikl}u_{l,kj} + \varepsilon_{jkl}u_{l,ki})$	Simplified model of Koiter's model. It can also be got from Kleinert's model by setting $l' = 0$
Mindlin and Tiersten (1962)	The Navier-Cauchy equation is the same as Koiter's: $\sigma_{ij,j} - l^2\nabla^2(u_{i,jj} - u_{j,ji}) = 0$	Essentially no different from Koiter's model which in turn can be obtained from Kelinert's formulation
Aifantis's model (2003—and references therein)	$\sigma_{ij} = (1 - l^2\nabla^2)\sigma^0_{ij}$	Aifantis' model can be got from Kleinert's by setting $l = l'$(i.e. ignoring couple stresses)
Fleck and Hutchinson (2001)	$W = \frac{1}{2}\lambda(\mathrm{tr}\,\varepsilon)^2 + \mu\varepsilon_{ij}\varepsilon_{ij} + l^2\chi_{ij}\chi_{ij}$	The same as Koiter's model, which can be obtained from Kleinert's model by $l = l'$

Appendix B. Determination of non-local parameters

In the context of elasticity, very few experiments have been conducted to determine non-local parameters (in elasticity much in contrast with strain gradient plasticity where numerous works can be cited) although a few works (including our own-unpublished) have employed atomistic methods and/or phonon dispersion curves to obtain these higher order parameters. Chen et al. (2003, 2004) have used phonon dispersion curves to obtain such constants. Shibutani et al. (1998) have also successfully obtained such properties through embedded atom method calculations. Our own atomistic calculations indicate that the characteristic length scale for non-local effects is generally to the order of the lattice parameter for most metals i.e. ~0.25 nm. In some systems it can be much higher (e.g. graphite = 3.7 nm, Reid and Gooding, 1992). In general non-local elastic terms are important for the following material systems: (i) Polymers (ii) granular materials (iii) composites—a recent work by Drugan (2000) clearly indicates that in general constitutive laws for composites are non-local in nature and well described by a strain gradient type formulation (iv) metals with defects under small stresses (i.e. gross plasticity does not take place but screening effects of dislocations cause non-local effects). See a recent paper by Sharma and Ganti (2004).

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